GGWeek 2022

17th European Research Week on Geometric Graphs

Campus Leipzig / FernUniversität in Hagen, Germany
August 15–19, 2022

Venue: Städtisches Kaufhaus, Universitätsstr. 16, 04109 Leipzig

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	Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
9:00 - 12:30		group work & coffee	group work & coffee	group work & coffee	progress reports
12:30 - 14:00	13:30: problem session	lunch	lunch	lunch	lunch
14:00 - 15:30		group work	group work	group work	
15:30 - 16:15	coffee	coffee	coffee	coffee	
16:15 - 18:00	group work	progress reports	social event	group work	

For Tuesday evening we made a reservation at Barthels Hof for dinner.

Problem 1 Connected Matching

suggested by Florian Thomas, Birgit Vogtenhuber, Oswin Aichholzer

Consider a (straight line) maching for a set of n points in the plane in general position. We say that two edges are *connected* (via their crossings) if they intersect. We want to know which size (number of edges) of a connected set we can guarantee for any given point set of size n. Figure 1(left) gives an example where the largest connected component has size $n/3$.

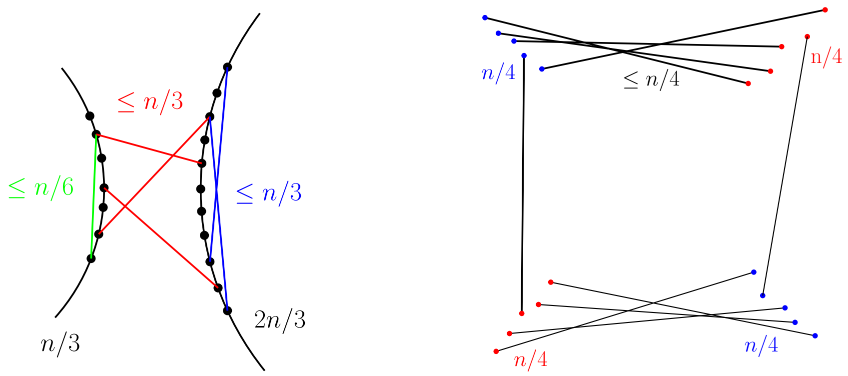

Figure 1: Upper bounds for (colored) connected matching edges.

In a variation of the problem $n/2$ points are colored red, and the other half is colored blue. We consider bicolored matchings, that is, an edge of the matching connects a red point to a blue point. Again we ask for the largest connected set of matching edges any given bicolored point set admints. Figure 1(right) gives an example of n points where the largest connected set has size $n/4$.

For both variants a linear lower bound with small constants (about $\frac{1}{32}$ for the uncolored case, worse for the colored case) can be shown. So the task is to find the right constant for the linear bound.

CONNECTED MATCHING
Input: Given n points, n even, in the plane in general position.
Question: What is the largest connected matching you can always guarantee?

BICOLORED CONNECTED MATCHING
Input: Given n points, n even, in the plane in general position, $n/2$ colored red, $n/2$ colored blue.
Question: What is the largest bicolored connected matching you can always guarantee?

Problem 2 Recognition of Squares

suggested by Till Miltzow

Given a bunch of geometric objects in the plane, we can define the intersection graph of those objects. Each object gives rise to a vertex and any two vertices are adjacent if the corresponding geometric objects intersect. If we limit the type of geometric objects to say: rays, grounded segments, segments, circles, unit disks, triangles, heart shapes, etc, we will get different graph classes. Given one of those classes C, the recognition question asks if a given graph G belongs to this class or not. For many geometric shapes we do know $\exists\mathbb{R}$-completeness for the recognition problem. To the best of my knowledge there is a gap in the literature for unit-square intersection graphs. It is reasonable to assume that the complexity of this problem is $\exists\mathbb{R}$-complete [3, 4, 5].

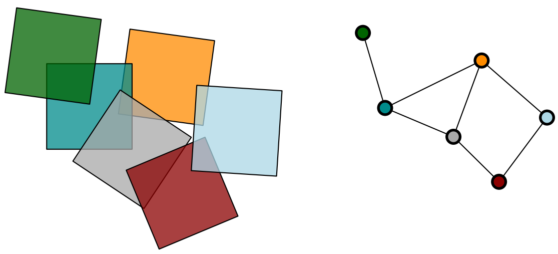

Figure 1: A set of unit squares in the plane gives rise to an intersection graph.

Question 1 *Is recognition of unit-squares graphs $\exists\mathbb{R}$-complete?*

One can then go to other shapes, like triangles, ellipses, squares, rectangles and many more geometric shapes. The natural conjecture is that the recognition problem for all of the corresponding geometric objects is $\exists\mathbb{R}$-complete. Exceptions are only made if the shapes are super simple, like axis parallel squares, or axis parallel segments [1]. Given a geometric shape S, we denote by C_S the corresponding intersection graph class.

Question 2 *For which S is the recognition problem for C_S $\exists\mathbb{R}$-complete?*

I expect that similar techniques on distinguishing geometric graph classes will be useful to study [2]. Having a general framework at least for convex shapes would indeed deepen, complete and solidify our intuitive understanding of geometric intersection graphs.

Remark 1 *I recently asked the question for unit-segment graphs and we found that the recognition problem is indeed* $\exists\mathbb{R}$*-complete.*

References

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Problem 3 Triangles in Arrangements

suggested by Manfred Scheucher

Grünbaum conjectured that every digon-free arrangement on n pairwise intersecting pseudocircles contains at least $p_3 \geq 2n-4$ triangles. Even though there are infinitely many arrangements that disprove his conjecture [1], all known examples contain a specific arrangement $\mathcal{N}_6^\Delta$ as subarrangement. Since $\mathcal{N}_6^\Delta$ has no representation with circles (two different proofs are given in [2]), Grünbaum's conjecture seems to be true for arrangements of proper circles.

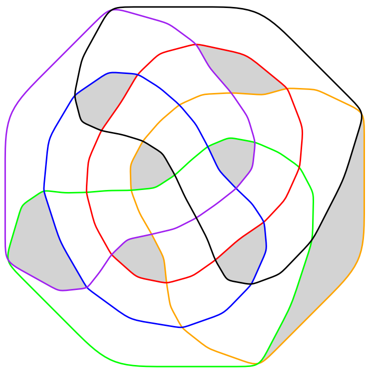

Figure 1: The non-circularizable arrangement $\mathcal{N}_6^\Delta$.

Several questions would be interesting in this context.

1. Can we show that every arrangement of pseudocircles without $\mathcal{N}_6^\Delta$ contain $2n-4$ triangles? (Is there a combinatorial proof?)
2. Can we show that every arrangement of proper circles contain $2n-4$ triangles? (For arrangements of lines there is a proof based on linear algebra to show $n-2$ triangles. Even though we managed to generalize this technique to circles, we so far only managed to show $\frac{n}{2}$ triangles, which is worse than the $\frac{4}{3}n$ bound by Snoeyink and Hershberger.)
3. Can we find a practical technique to certify non-circularizability such as the method of biquadratic final polynomial by Richter-

Gebert? (So far, all non-circularizability proofs are tailored to the respective arrangement.)

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[1] Stefan Felsner and Manfred Scheucher. Arrangements of Pseudocircles: Triangles and Drawings. In Discrete & Computational Geometry 65, pages 261–278, 2021.

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Problem 4 Maximum Matchings in Yao- and Θ-graphs

suggested by Philipp Kindermann

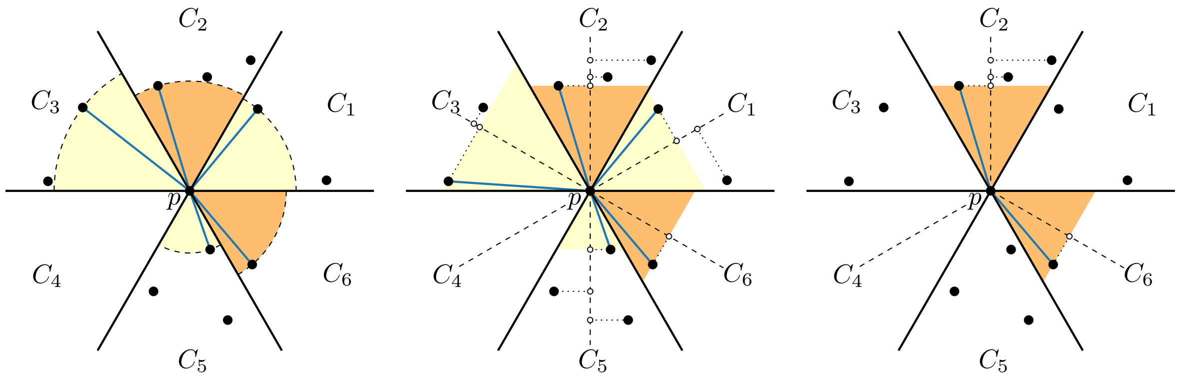

Figure 1: Construction of a (left) Y_6-graph; (middle) Θ_6-graph; and (right) Half-Θ_6-graph.

The Yao-graph Y_k on a set P of points in the plane is a geometric graph with vertex set P and edges constructed as follows [6]. For every point $p \in P$, place k rays emanating from p at angles that are multiples of $2\pi/k$ radians from the positive x-axis. These rays partition the plane into k cones $C_1, \dots, C_k$ with apex p; see Fig. 1. Add an edge from p to the *closest* point in each cone C_i, measured by its Euclidean distance to p. The Θ_k-graph on P is defined equivalently, but the distance between the apex p and a point q in C_i is measured by the Euclidean distance from p to the projection of q on the bisector of C_i [3, 5]. The *Half-Θ_k-graph* on a point set P is the subgraph of the Θ_k-graph composed only of the even-cone edges.

Yao- and Θ-graphs have many nice properties, e.g., they are sparse for constant k, they are connected for $k \geq 2$, and they have constant spanning ratio for $k \geq 4$. Half-Θ_6-graphs are equivalent to triangular-distance Delaunay graphs.

Every Delaunay triangulation contains a (near-)perfect matching, i.e., at most one point is unmatched [4]. For Yao- and Θ-graphs, equivalent results are largely unknown. Babu et al [1] conjectured that every Θ_6-graph has a (near-)perfect matching, but I'm only aware of the following two results.

Theorem 1 ([1]) *Every Half-Θ_6-graph on n points has a matching of size $\lceil (n-1)/3 \rceil \approx n/3$, and this bound is tight.*

Theorem 2 ([2]) *Every Θ_6-graph on n points has a matching of size $\lceil (3n-8)/7 \rceil \approx 3n/7$.*

I'm mainly interested in the following questions, but any variant is fine as well.

Question 1 *Does every Θ_4-graph on n points have a matching of size $cn - O(1)$ with $c > 0$?*

Question 2 *Does every Θ_6-graph on n points have a matching of size $cn - O(1)$ with $c > 3/7$?*

Question 3 *Does every Y_6-graph on n points have a matching of size $cn - O(1)$ with $c > 0$?*

Question 4 *Does every Y_k- or Θ_k-graph on n points have a (near-)perfect matching for some $k \geq 6$?*

MAXIMUM MATCHINGS IN YAO- AND Θ-GRAPHS
Input: A set P of n points in the plane, and an integer $k > 0$.
Question: What is the largest matching that we can guarantee in Y_k or Θ_k?

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Problem 5 Straight Line Drawings of Planar Graphs

suggested by Antonia Kalb

A drawing is a *deformation* of another drawing if the positions of the vertices differ, but not the faces with which they are incident (see Figure 1). We want to know if there are (*k*-regular) planar drawings which has always a deformation fulfilling certain properties.

For our research (see Remark 1) we are interested in the following properties: The outer face of a drawing is *convex*, if for any three clockwise following vertices incident to the outer face the outer angle is $> 180°$. We call an inner face "zigzag"-shaped if the face is bounded by two "interleaving" chains of alternating acute and reflex angles (see Figure 2). If done properly, sub graphs can be placed such that they do not *see* each other (edges to connect the sub graphs intersect other edges).

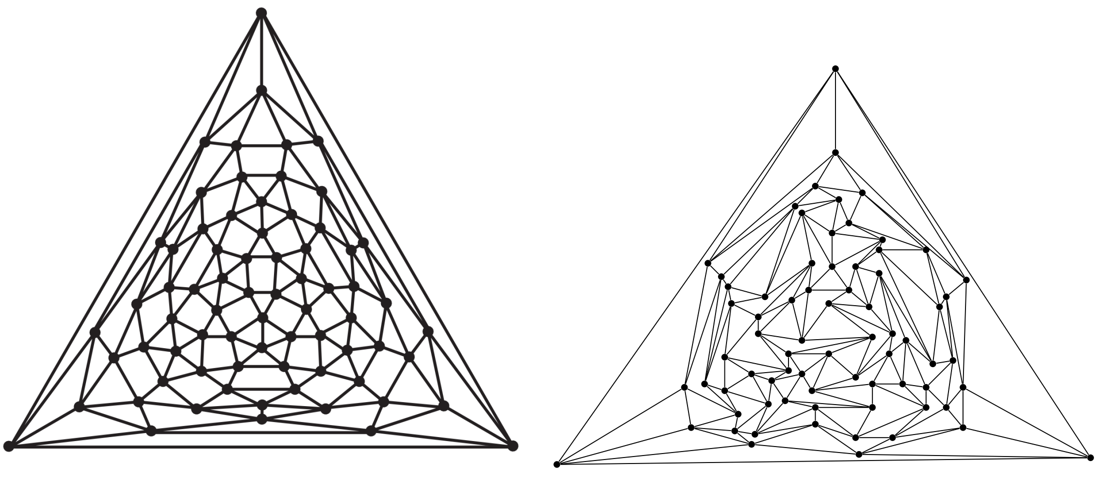

Figure 1: A planar drawing of a 5-regular graph and its deformation such that the outer face is convex and ever inner face is "zigzag"-shaped

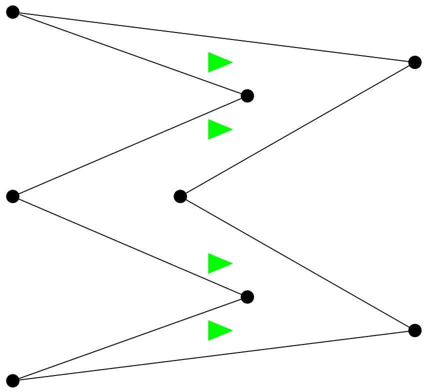

Figure 2: An inner "zigzag"-shaped face, the extended sub graphs (green) can not connected by edges without intersections

DEFINING PROPERTIES
Input: Drawing of a connected k-regular planar graph for $k \geq 2$
Question: Are there some drawings which always have a deformation with a convex outer face and only "zigzag"-shaped inner faces?

For example, a planar drawing where a tour on the vertices of the outer face visits a vertex twice, cannot be deformed such that the outer face is convex (see Figure 3).

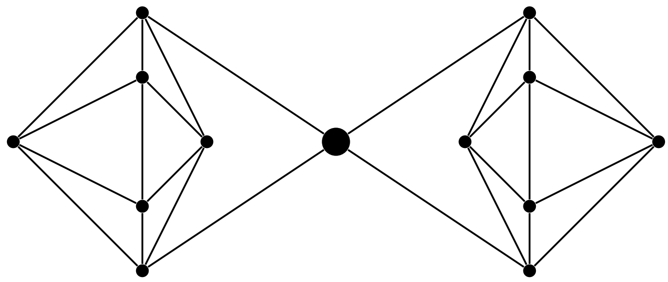

Figure 3: Drawing of a 4-regular graph which has no deformation with a convex outer face

DEVELOP ALGORITHM
Input: Drawing of a connected k-regular planar graph for $k \geq 2$
Question: Can we develop an algorithm which deforms the drawing such that the outer face is convex and the inner faces are "zigzag"-shaped?

Remark 1 (Motivation) *A matching M is an edge set that is disjoint to the edges of a graph and M is compatible if it preserves the planarity of the graphs fixed straight-line drawing augmented by M. A matching is maximal if it cannot be expanded to a larger compatible matching. For* 0-, 1- *and* 2-*regular graphs, Pilz et al. [3] proved tight lower bounds for the size of the minimal maximal compatible matching depending on number of vertices. We [1, 2] proved lower bounds for* 3- *and* 4-*regular graphs, but still search for drawings to prove the tightness of our bounds. We constructed drawings which reach very low matching sizes and a formula to compute their matching size without actually drawing them. This formula could be used to search automatically for a tight drawing if we know that any automatically generated graph holds the properties we defined in our construction. These properties are a convex outer face and only "zigzag"-shaped inner faces. The bounds for* 0-, 1- *and* 2-*regular graphs by [3] are tight due to graphs that correspond to this construction.*

References

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Problem 6 Snapping graphs to the grid

suggested by Antonia Kalb

Snap rounding is a assignment of vertices of a planar graph to integer point in $\mathbb{N}^2$, such that the resulting *straight-line grid drawing* is planar. For line segments, lots of fast algorithms exists [5, 6, 7, 8]. However, these algorithms are not topologically safe: vertices merge, edges bend or even faces can disappear while rounding. De Berg et al. [1] developed a sensitive algorithm for line arrangements. For graphs, minimizing the area or number of needed grid points is studied well [2, 3, 4, 9].

TOPOLOGIALLY SAFE SNAPPING
Input: Planar graph $G = (V, E)$ with real vertex positions $r : V \mapsto \mathbb{R}^2$
Question: Vertex positions $p : V \mapsto \mathbb{N}^2$, such that the drawing $\Pi(G, p)$ is topologically equivalent to drawing $\Pi(G, r)$, and $\sum_{v \in V} |r(v) - p(v)|$ (Sum of Euclidean distances) is minimized

Löffler et al. [10, 11] proved NP-Hardness for TOPOLOGIALLY SAFE SNAPPING and define an ILP. In [10] the approximability of the problem is discussed and they prove that no fully polynomial-time approximation scheme and no constant additive approximation exists. But they were not able to fix the classification.

Question 1 *What is the general approximability of* TOPOLOGIALLY SAFE SNAPPING*?*

Löffler et al. [10, 11, 12] designed heuristic approaches and evaluate these on random graphs.

Question 2 *Can we design approximation algorithms for* TOPOLOGIALLY SAFE SNAPPING*? What guarantees can we provide?*

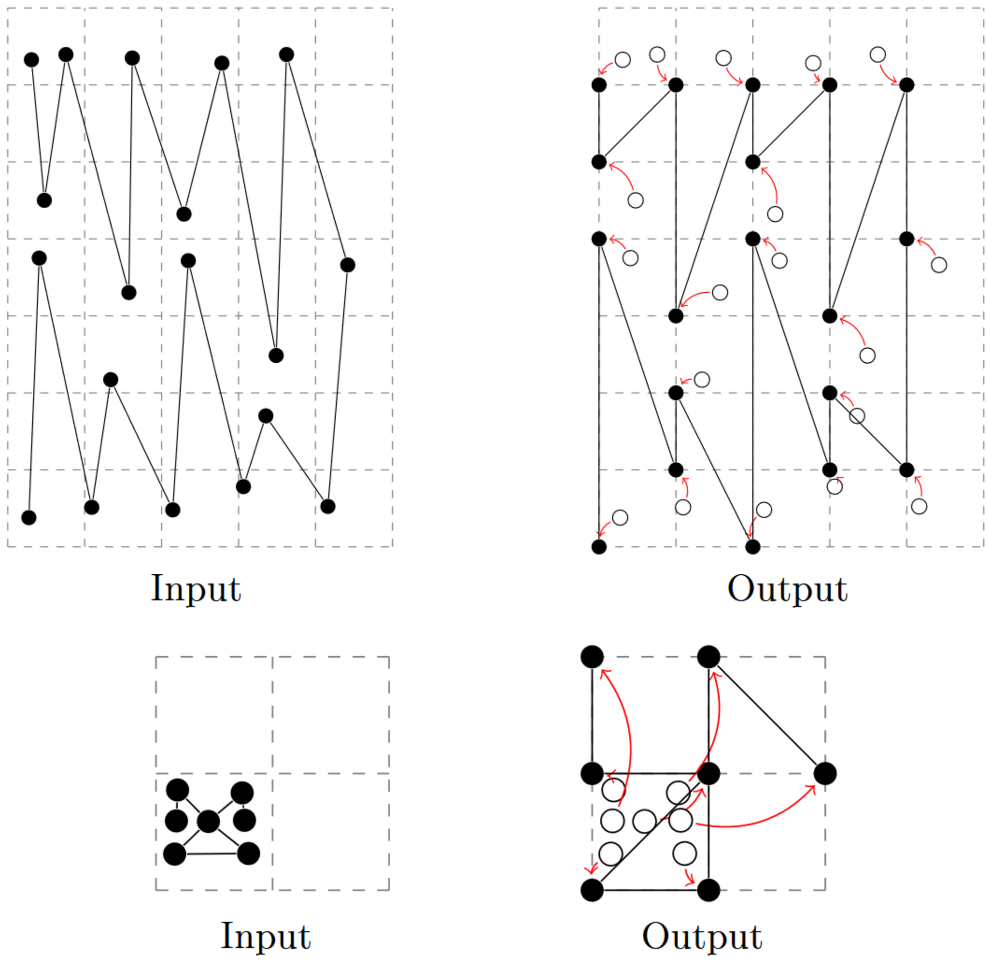

Figure 1: Some examples of snap roundings, taken from [11]

References

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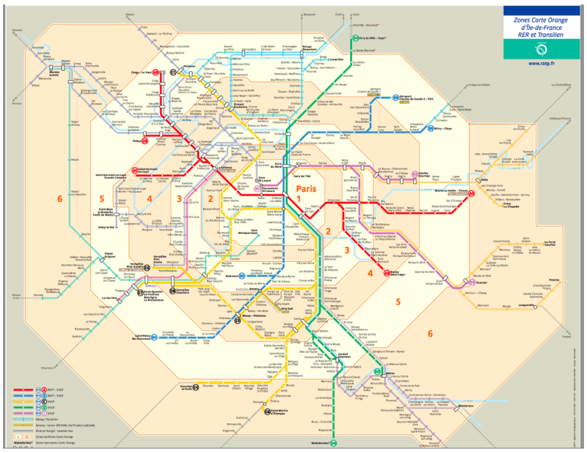
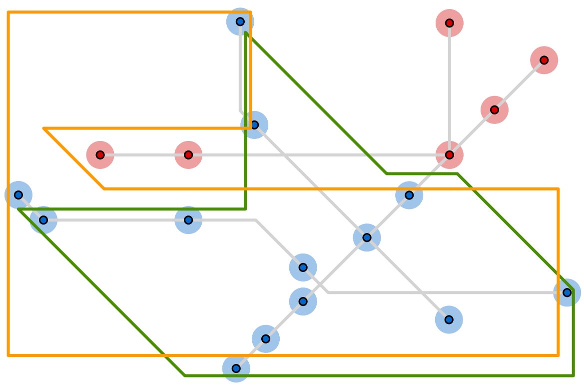
Problem 7 Maps with Tariff Zones

suggested by Soeren Terziadis

The motivation of this problem comes from schematic mapping. Often we want to overlay a schematic graph drawing with a schematized (in this case C-oriented, see Figure 1a) polygon to indicate tariff zones, areas reachable within a certain time, zoning, etc.. Such a polygon will include some elements (usually vertices) of the graph, and exclude others. Since we are working within a schematized setting, we use the complexity of the polygon, i.e., the number of edges on its boundary to measure its length (instead of euclidean distance). We call such a polygon a minimum link polygon. So the most straight-forward question would be to find a C-oriented minimum link polygon P, which includes certain vertices and excludes others.



(a)



(b)

Figure 1: (a) Map of the metro of Paris with C-oriented tariff zones, i.e., all edges are parallel to one of $|C|$ orientations. (b) Instance with unit disks. The orange and green polygons are solutions of size 8. The orange polygon additionally does not include or cross any edge between two red vertices.

This would be used in a two step approach of computing the drawing first and then computing P. However, there might be some simple adjustments to the drawing, which might lead to a better solution for P. To model this, we represent the acceptable area of a vertex as a shape (unit-disk, square, diamon, etc.). The assumption is that any position in this area is an acceptable placement of the vertex, independent of the placement of any other vertex. An instance with two solutions is sketched in Figure 1b.

Minimum Link C-Oriented Hitting Polygon
Input: Two sets of unit disks $\mathcal{A}$ and $\mathcal{B}$
Question: Can we find a minimum link polygon P, s.t., $\forall A \in \mathcal{A} \exists a \in A : a \in P \wedge \forall B \in \mathcal{B} \forall b \in B : p \notin P$? Or more verbose, P includes at least one point of every A and no point of any B.

Computation of minimum link paths and loop polygons has received some attention and is related to polygon nesting. Aggarwal et al. [2] and later Wang [6] present algorithms, which compute a minimum link polygon surrounding an inner polygon, while being contained in an outer polygon. Hershberger and Snoeyink [5] presented an algorithm to compute C-oriented paths and loops within a given C-oriented polygon with holes, for a predefined homotopy in polynomial time. Adegeest et al. [1] investigated C-oriented minimum link paths among obstacles and present a data structure, which allows for $O(n \log n)$ construction time of such a path. Recent results include Baum et al. [3] present a new linear time heuristic to compute minimum link paths within a given polygon, which they use to compute minimum link isocontours as an overlay of road networks. Bonerath et al. [4] compute "tight" C-oriented hulls of a given polygon, which can not be shrunk in any way, without overlapping the containing polygons. The key difference is that all polygons need to be completely contained.

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Problem 8 Realizability of arc-polygons

suggested by André Schulz

A *(circular) arc-polygon* is a closed curve with no self-intersections which is assembled by a finite sequence of circular arcs. See Figure 1 for some examples. We measure the (interior) angle at a vertex (the point where two arcs meet) as the angle between the tangents in the interior.

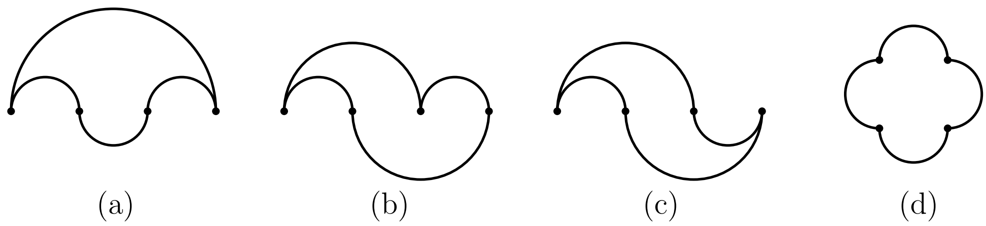

Figure 1: Four arc-polygons with angle sequences (a):$(0, \pi, \pi, 0)$, (b):$(0, \pi, \pi, 2\pi)$, (c):$(0, \pi, 0, \pi)$, (d):$(3\pi/2, 3\pi/2, 3\pi/2, 3\pi/2)$.

Eppstein et al. studied the question for which sequences of angles a realization as an arc-polygon exists. The motivation for this question stems from the question which graphs have a planar Lombardi drawing (edges are circular arcs, no crossings, and the angles around a vertex v are all of size $2\pi/\deg(v)$). In a recent paper it is shown that cactus graphs do always have an embedding, which can be drawn as a planar Lombardi drawing but there are also embeddings that cannot be drawn as planar Lombardi drawings [1]. These results were obtained from a complete characterization of realizable arc-triangles: Eppstein et al. proved that an arc-triangle is realizable, iff for $i = 1, 2, 3$ we have $-\pi < \phi_i < \pi$, where ϕ_i is the angle as depicted in Figure 2.

A complete characterization for general arc-polygons is not known. However Eppstein et al. provided some partial results.

- If the angular sequence is alternating between 0 and 2π and is furthermore even it has no realization as an arc-polygon [1].
- If all angles θ_i are within $0 \leq \theta_i \leq \pi$ then all sequences can be realized as an arc-polygon with the only exception of $(0, 0, \pi)$ [1].

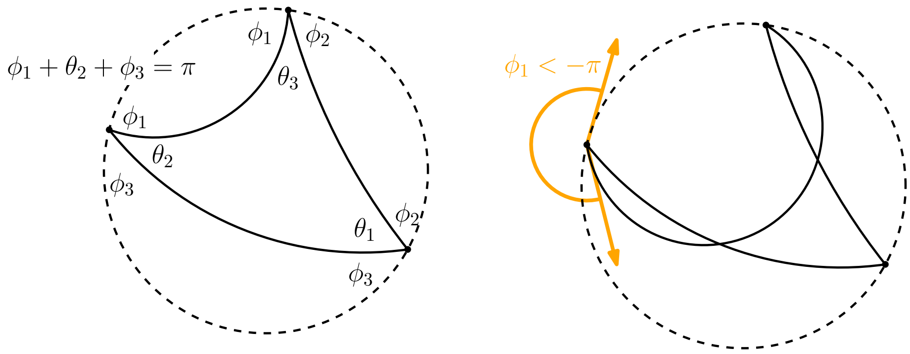

Figure 2: The angles ϕ_i at an arc-triangle. Measurements are taking with respect to the circle passing through the three vertices of the arc-triangle. Algebraically: $\phi_i = \theta_i + (\pi - \sum \theta_i)/2$.

- For the sequence $(\pi/2, \pi/2, \pi/2, \pi/2)$ all vertices have to lie on a common circle that does not intersect the arc-polygon [2].

Question 1 *Can we give a (more) complete characterization of the angular sequences that do have a realization as an arc-polygon.*

Even a complete charcterization for arc-quadrilaterals would be an interesting result. One could start with the special case that the 4 points have to lie on a common circle. A question that might be related is also the following.

Question 2 *Can we triangulate every arc-polygon with circular arcs.*

References

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Problem 9 Queue Number of Planar Laman Graphs

suggested by Jonathan Rollin

An n-vertex graph G is called a *Laman graph* if it has exactly $2n-3$ edges and every subset $V' \subseteq V(G)$, with $|V'| \geq 2$, induces a subgraph with no more than $2|V'|-3$ edges. Figure 1(left) shows an example of a Laman graph. Laman graphs come with many different characterizations and are of particular interest in the theory of rigid graphs. Among other results, it turned out that the planar Laman graphs form an interesting subclass of the family of all planar graphs [1].

A linear layout of a graph consists of a total ordering of the vertices and a partition of the edge set satisfying various constraints. Specifically, the queue number $\mathrm{qn}(G)$ of a graph G is defined as the smallest number k such that there is a linear layout of G where the edges are partitioned in to k sets $E_1, \ldots, E_k$ such that each set does not contain two nested edges. Here, two edges are called nested, if they are independent and, in the vertex ordering, the endpoints of one edge are in between the endpoints of the other edge; see Figure 1 (middle, right) for an illustration.

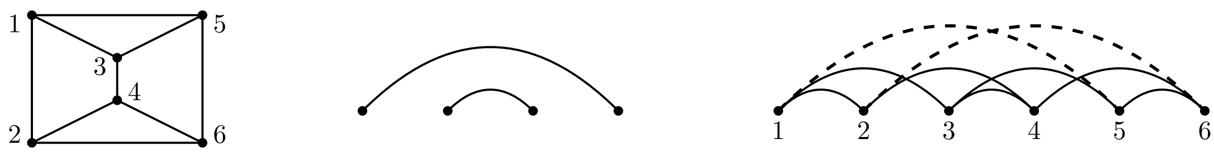

Figure 1: Left: A planar Laman graph Γ. Middle: Two nested edges. Right: A 2-queue layout of Γ with edge partition (solid and dashed).

Recently, linear layouts of planar graphs received considerable attention and substantial progress on several open questions has been made. For instance, the current best bounds on the queue number of planar graphs are $4 \leq \mathrm{qn}(G) \leq 42$ [2, 3]. We are interested in linear layouts of planar Laman graphs. The best upper bound on the queue number of a planar Laman graph G comes from the bound on general planar graphs stated above while a lower bound $\mathrm{qn}(G) \geq 2$ holds for all so-called X-trees G [4], which are Laman graphs.

Question 1 *What is the largest queue number among all planar Laman graphs?*

There are many ways to attack this question via the different characterizations of (planar) Laman graphs. Note that general (not necessarily planar) Laman graphs contain all full 1-subdivisions of complete graphs as subgraphs and hence their queue number is unbounded [5].

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Problem 10 Saturated k-planar Drawings

suggested by Jonathan Rollin

A drawing of a graph is k-planar if each edge of the graph is crossed at most k times. These drawings constitute a classical family of beyond planar graphs and received considerable attention. A fundamental open question regards the largest number of edges among all k-planar drawings with tight bounds known only for small values of k. Each drawing with the largest number of edges is clearly saturated, which means that adding any edge to the drawing violates k-planarity. Interestingly, there are also much sparser saturated k-planar drawings [1, 2, 3].

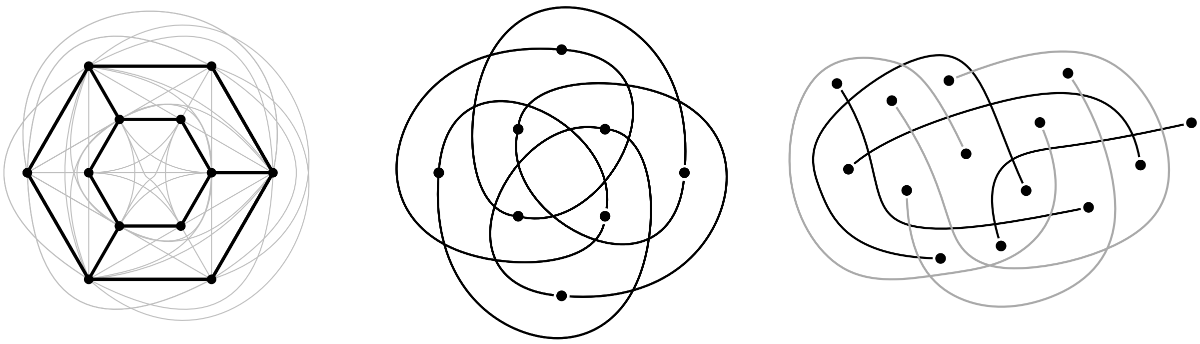

Figure 1: From left to right: A 4-planar drawing with $6n - 12$ edges, a 4-planar drawing of a cycle, a 6-planar drawing of a matching. All drawings are saturated (if loops and parallel edges are not allowed).

We are interested in the question which graphs admit a saturated k-planar drawing for some k. To work on this question it is necessary to decide which type of drawing is considered. Most results on k-planar drawings consider simple drawings, that is, drawings where any two edges intersect at most once (in other words: loops, parallel edges, selfcrossing edges, crossing incident edges, and edges crossing more than once are not allowed). But also non-simple drawings are interesting and have been studied as well [3, 4]. To work with different types of drawings we define so-called drawing styles, which are just classes of drawings, e.g., the class of all k-planar simple drawings forms one particular drawing style.

Saturated *k*-planar Drawings
Input: Given a drawing style Γ.
Question: Determine for which graphs *G* there is some *k* such that *G* admits a saturated *k*-planar drawing in Γ.

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Problem 11 2-Colored Crossing Number

suggested by Birgit Vogtenhuber

Let S be a set of n points in the plane in general position, that is, no three points are on a line. Consider the straight-line drawing D of the complete graph K_n which has the points of S as vertices. We want to color the edges of D with two colors in a way that the number of monochromatic crossings (that is, crossings between edges of the same color) is minimized. We are interested in how efficiently such a coloring can be computed.

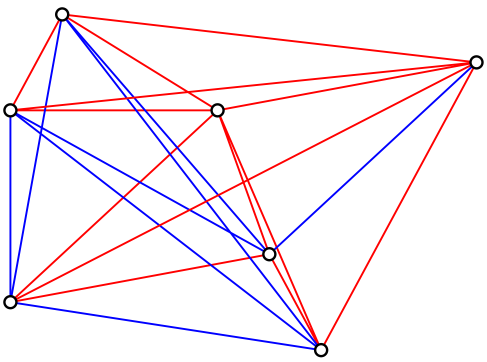

Figure 1: A drawing of K_7 with a non-optimal edge-coloring and 6 monochromatic crossings.

Definition 1 *Let D be a (straight-line) drawing of a graph G (on top of a point set S). The* 2-colored crossing number $\mathrm{cr}_2(D)$ *of D is the minimum over all edge* 2*-colorings $\chi(D)$ of the number of monochromatic crossings in $\chi(D)$.*

COMPUTING THE 2-COLORED CROSSING NUMBER $\mathrm{cr}_2(D)$ **Input:** A straight-line drawing D of K_n **Question:** How fast can we compute 2-coloring of D that has only $\mathrm{cr}_2(D)$ monochromatic crossings?

Remark 1 *If S is in convex position then an optimal 2-coloring of the straight-line drawing of K_n on S can be efficiently computed [1].*

Remark 2 *If instead of the complete graph, we are given a straight-line draw-*

ing of a general graph G on a point set S, then the problem of deciding whether $\mathrm{cr}_2(D) \leq k$ *is NP-hard, even if S is in convex position [2].*

COMPLEXITY OF DECIDING $\mathrm{cr}_2(D) \leq k$
Input: A straight-line drawing D of K_n and an integer k
Question: What is the compexity of deciding whether $\mathrm{cr}_2(D) \leq k$?

Remark 3 *The question is identical to finding a maximum cut in the* crossing graph, *that is, the intersection graph of the edges of D where enpoints are not part of the edges.*

Question 1 *What if we allow a constant number* $c > 2$ *of colors?*

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